Special entangled quantum systems and the Freudenthal construction

2009 J. Phys. A: Math. Theor. 42 285303

(http://iopscience.iop.org/1751-8121/42/28/285303)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 285303 (18pp) doi:10.1088/1751-8113/42/28/285303

Special entangled quantum systems and the Freudenthal construction

Péter Vrana and Péter Lévay

Department of Theoretical Physics, Institute of Physics, Budapest University of Technology and Economics, H-1521 Budapest, Hungary

Received 17 March 2009, in final form 21 May 2009
Published 19 June 2009
Online at stacks.iop.org/JPhysA/42/285303

Abstract
We consider special quantum systems containing both distinguishable and identical constituents. It is shown that for these systems the Freudenthal construction based on cubic Jordan algebras naturally defines entanglement measures invariant under the group of stochastic local operations and classical communication (SLOCC). For this type of multipartite entanglement the SLOCC classes can be explicitly given. These results enable further explicit constructions of multiqubit entanglement measures for distinguishable constituents by embedding them into identical fermionic ones. We also prove that the Plücker relations for the embedding system provide a sufficient and necessary condition for the separability of the embedded one. We argue that this embedding procedure can be regarded as a convenient representation for quantum systems of particles which are *really* indistinguishable but for some reason they are not in the same state of some inner degree of freedom.

PACS numbers: 02.40.Dr, 03.65.Ud, 03.65.Ta, 03.67.−a

1. Introduction

It has become common wisdom that quantum entanglement can successfully be regarded as a resource for performing different tasks connected with the processing of quantum information. However, in order to fully exploit further exciting possibilities still hidden in this resource it has to be carefully quantified and its different types classified. The desire to achieve this goal has initiated a detailed mathematical study on the construction and structure of entanglement measures with particular physical relevance. Such measures (or witnesses) are real-valued functions on pure or mixed entangled states capable of grasping a particular aspect of quantum entanglement of the physical system such states represent. For a subclass of entanglement measures, the basic property is their invariance under a particular group of transformations representing admissible local operations on the entangled subsystems of the physical system. Two important classes of such transformations are the group of local unitary transformations (LU) and the so-called SLOCC group [1] corresponding to stochastic local operations and classical communication. For the LU and the SLOCC classification of systems of *distinguishable* constituents characterized by either pure or mixed states a great variety of results is available [2–4].

However, much less is known about the structure of multipartite entanglement measures and the corresponding entanglement classes for systems with *indistinguishable* constituents. For *bipartite* fermionic and bosonic systems, a number of useful results exist [5–12] related to the existence of a variant of the conventional Schmidt decomposition. For the classification of multipartite entanglement of fermionic and bosonic sytems, Eckert *et al* [5] provided an analysis with some hints on how useful entanglement measures should be constructed. As a next step in our recent paper, we have shown that for tripartite fermionic systems with six single-particle states a genuine measure of tripartite entanglement exists, and the corresponding SLOCC classification can be fully given [13]. The striking feature of this classification is its similarity with the corresponding one found for three qubits [1]. We have shown that this correspondence between a tripartite system with identical and a tripartite one with indistinguishable constituents has its roots in their common underlying mathematical structure related to Freudenthal systems based on cubic Jordan algebras [14–16].

Freudenthal triple systems have already made their debut to physics within the realm of string theory and supergravity. Such systems can be used as a representation of the charge vector space occurring when studying certain black hole solutions in $N = 8$, $d = 4$ supergravity [17–21]. Recently, striking multiple relations have been established between the physics of such stringy black hole solutions and quantum information theory [22–30, 32]. Though this 'black hole analogy' still begs for a physical basis, the underlying correspondences have repeatedly proved to be useful for obtaining new insights into one of these fields by exploiting the methods established within the other. Since Freudenthal triples have already turned out to be important in the string theoretic context the idea is to use this algebraic structure also in quantum information [31, 32]. A latest variation on this theme that appeared in the literature is a Freudenthal triple based reconsideration of three-qubit entanglement [33]. Following this trend in this paper, we extend the range of applications of Freudenthal systems also to include entangled systems containing *both* distinguishable and indistinguishable constituents. The basic idea of our investigation is that of embedding one type of system into another one. Since Freudenthal systems are too special retaining merely the idea of embedding from them, in the second half of the paper we explore this principle in a more general context.

The plan of the paper is as follows. In section 2, we consider Freudenthal systems giving rise to combined systems containing bosonic and fermionic constituents. In section 3, after reconsidering the case of three fermions with six single-particle states, we discuss the systems containing one distinguished qubit and two bosonic qubits, three bosonic qubits, and one with one qubit and two fermions with four single-particle states. Then we give the representatives of SLOCC equivalence classes. As a generalization in section 4, we discuss systems with distinguishable constituents that can be embedded into fermionic ones. We investigate issues of separability, and their relation to the Plücker relations well known from multilinear algebra. Then we concentrate on a phenomenon called the splitting of SLOCC classes, a method which might prove to be a useful tool in providing entanglement classification in more complicated systems. As a next step, we introduce a family of SLOCC invariants for fermionic systems. We conclude with an explicit calculation of reduced density matrices for our embedded systems. A possible physical interpretation of our embedding procedure is briefly mentioned.

2. Cubic Jordan algebras and Freudenthal triples

Definition. *An algebra (not necessarily associative)* $(J, +, \bullet)$ *is called a Jordan algebra if for any two elements* $A, B \in J$ *the equations*

$$A \bullet B = B \bullet A \tag{1}$$

and

$$(A \bullet A) \bullet (A \bullet B) = A \bullet ((A \bullet A) \bullet B) \tag{2}$$

hold.

A Jordan algebra is cubic if every element satisfies a cubic polynomial equation.

The Springer construction of cubic Jordan algebras tells us that one can obtain a cubic Jordan algebra starting with a vector space V equipped with a suitable cubic form $N : V \to \mathbb{C}$ and a basepoint $c \in V$ such that $N(c) = 1$. One can then define various maps using the linearization,

$$\begin{aligned} N(x, y, z) = \tfrac{1}{6}(N(x + y + z) - N(x + y) \\ - N(x + z) - N(y + z) + N(x) + N(y) + N(z)), \end{aligned} \tag{3}$$

of N, including the Jordan product, but for our purposes only the following two are needed:

$$(\cdot, \cdot) : V \times V \to \mathbb{C}; \qquad (x, y) = 9N(c, c, x)N(c, c, y) - 6N(x, y, c), \tag{4}$$

$$\cdot^\sharp : V \to V; \qquad \forall\, y \in J : (x^\sharp, y) = 3N(x, x, y). \tag{5}$$

The former is called the trace bilinear form, while the latter is the adjoint or sharp map.

From a Jordan algebra J over $\mathbb{C}$ one can obtain the Freudenthal triple system, $\mathfrak{M}(J) = \mathbb{C} \oplus \mathbb{C} \oplus J \oplus J$, which is equipped with a skew-symmetric bilinear form and a quartic form defined by

$$\{x, y\} = \alpha\delta - \beta\gamma + (A, D) - (B, C), \tag{6}$$

$$q(x) = 2((A, B) - \alpha\beta)^2 - 8(A^\sharp, B^\sharp) + 8\alpha N(A) + 8\beta N(B), \tag{7}$$

where $x = (\alpha, \beta, A, B)$ and $y = (\gamma, \delta, C, D)$ are two elements of $\mathfrak{M}(J)$. One can also define the unique trilinear map, $T : \mathfrak{M}(J) \times \mathfrak{M}(J) \times \mathfrak{M}(J) \to \mathfrak{M}(J)$, with the property $\{T(x, y, z), w\} = q(x, y, z, w)$ where $q(\cdot, \cdot, \cdot, \cdot)$ is the linearization of the quartic form $q(\cdot)$.

Definition. $\mathrm{Inv}(\mathfrak{M}(J))$ *is the group of linear transformations which preserve these forms, i.e. for all* $\sigma \in \mathrm{Inv}(\mathfrak{M}(J))$

$$\{\sigma(\cdot), \sigma(\cdot)\} = \{\cdot, \cdot\} \qquad \textit{and} \qquad q \circ \sigma = q \tag{8}$$

holds.

Clearly, the construction yields a $2 + 2\dim J$-dimensional representation of $\mathrm{Inv}(\mathfrak{M}(J))$ and q is a quartic polynomial invariant under the action of this group. In the following, we give explicitly the Jordan algebras needed for the classification of entangled states in the above-mentioned quantum systems. It turns out that all of them can be regarded as a subalgebra of $J_3 = M(3, \mathbb{C})$ so we start with this one.

For $A \in J_3$ the cubic form N is simply the determinant of the 3×3 matrix, for $A, B \in J_3$ the trace bilinear form is given by $(A, B) = \mathrm{Tr}(AB)$, and the explicit form of the sharp map is

$$A^\sharp = A^2 - \mathrm{Tr}(A)A + \tfrac{1}{2}(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))I_3. \tag{9}$$

The simplest nontrivial Jordan algebra is $J_1 = \mathbb{C}$, the cubic norm of $a \in J_1$ is $N(a) = a^3$. It follows that the map $J_1 \to J_3$; $a \mapsto aI_3$ is an injective morphism of cubic Jordan algebras. The next Jordan algebra is $J_{1+1} = \mathbb{C} \oplus \mathbb{C}$; the value of N on the element $x = (a, b)$ is $N(x) = ab^2$. In this case, the image of x in J_3 is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}. \tag{10}$$

The third Jordan algebra is $J_{1+1+1} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Here for $x = (a, b, c)$ the value of N is $N(x) = abc$. This is nothing else but the determinant of the matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{11}$$

which shows us the isomorphism between J_{1+1+1} and the subalgebra of diagonal matrices in J_3. The last Jordan algebra we consider is $J_{1+2} = \mathbb{C} \oplus Q_4$, where Q_4 is any four-dimensional complex vector space with a nondegenerate quadratic form. It is convenient to let Q_4 be the vector space of 2×2 matrices, and the quadratic form be the determinant. A general element in J_{1+2} is therefore $x = (a, A)$, and its cubic norm is $N(x) = a \det A$. For this Jordan algebra the inclusion map is given by

$$\left(a, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \mapsto \begin{bmatrix} a & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix}, \tag{12}$$

the image being a block diagonal matrix built from a 1×1 and a 2×2 block.

What makes these constructions useful for studying entanglement is the fact that the Inv groups are almost SLOCC groups of various quantum systems. Namely, $\mathrm{Inv}(\mathfrak{M}(J_1)) \simeq SL(2, \mathbb{C})$, $\mathrm{Inv}(\mathfrak{M}(J_{1+1})) \simeq SL(2, \mathbb{C})^2$, $\mathrm{Inv}(\mathfrak{M}(J_{1+1+1})) \simeq SL(2, \mathbb{C})^3$, $\mathrm{Inv}(\mathfrak{M}(J_{1+2}))$ is isomorphic to $SL(2, \mathbb{C}) \times SL(4, \mathbb{C})$ and finally $\mathrm{Inv}(\mathfrak{M}(J_3)) \simeq SL(6, \mathbb{C})$. The SLOCC groups are obtained by replacing $SL(n, \mathbb{C})$ with $GL(n, \mathbb{C})$. Excluding the zero vector we can identify four SLOCC orbits (characterized by the rank of vectors) in each case except the first one where the rank 2 orbit is absent. Vectors of rank 4 are those for which q does not vanish; all others are of rank at most 3. x has rank 3 iff $q(x) = 0$ and $T(x, x, x) \neq 0$. Vectors with rank 2 are those for which $T(x, x, x)$ vanishes but there exists y such that $3T(x, x, y) + \{x, y\}x \neq 0$. If there is no such y then x has rank 1.

3. Application to tripartite quantum systems

In our recent paper [13], we have shown that a genuine tripartite entanglement measure for three fermions with six single-particle states can be constructed using Freudenthal's construction applied to the cubic Jordan algebra of $J_3 = M(3, \mathbb{C})$. As a starting point in this section we recap our results.

It can be shown that the group of transformations of the Freudenthal triple system, $\mathfrak{M} = \mathbb{C} \oplus \mathbb{C} \oplus J_3 \oplus J_3$, preserving its quartic form is precisely $SL(6, \mathbb{C})$ and the representation of this group on $\mathfrak{M}$ is isomorphic to $\bigwedge^3 V_6$ where V_n denotes the standard representation of $SL(n, \mathbb{C})$ on the vector space of n-tuples of complex numbers. This group is a subgroup of the SLOCC [1] group i.e. $GL(6, \mathbb{C})$ acting for three fermions with six single-particle states.

An isomorphism clarifying such issues can explicitly be given as follows. Let $\{e_1, e_2, e_3, e_{\bar{1}} \equiv e_4, e_{\bar{2}} \equiv e_5, e_{\bar{3}} \equiv e_6\}$ be an orthonormal basis of $\mathbb{C}^6$, and let $x \wedge y \wedge z$ denote the normalized wedge product of the vectors $x, y, z \in \mathbb{C}^6$:

$$x \wedge y \wedge z = \frac{1}{\sqrt{6}}(x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y - x \otimes z \otimes y - z \otimes y \otimes x - y \otimes x \otimes z). \tag{13}$$

Using these notations a three-fermion state may be written as

$$P = \sum_{1 \leqslant a<b<c \leqslant \bar{3}} P_{abc} e_a \wedge e_b \wedge e_c \tag{14}$$

with 20 independent coefficients satisfying the condition

$$\sum_{1 \leqslant a<b<c \leqslant \bar{3}} |P_{abc}|^2 = 1, \tag{15}$$

meaning that the norm of the state is 1. The corresponding element of $\mathfrak{M}$ is $x = (\alpha, \beta, A, B)$, where

$$\alpha = P_{123} \quad \beta = P_{\bar{1}\bar{2}\bar{3}} \quad A = \begin{bmatrix} P_{1\bar{2}\bar{3}} & P_{1\bar{3}\bar{1}} & P_{1\bar{1}\bar{2}} \\ P_{2\bar{2}\bar{3}} & P_{2\bar{3}\bar{1}} & P_{2\bar{1}\bar{2}} \\ P_{3\bar{2}\bar{3}} & P_{3\bar{3}\bar{1}} & P_{3\bar{1}\bar{2}} \end{bmatrix} \quad B = \begin{bmatrix} P_{\bar{1}23} & P_{\bar{1}31} & P_{\bar{1}12} \\ P_{\bar{2}23} & P_{\bar{2}31} & P_{\bar{2}12} \\ P_{\bar{3}23} & P_{\bar{3}31} & P_{\bar{3}12} \end{bmatrix}. \tag{16}$$

Then the quartic polynomial preserved by the action of $SL(6, \mathbb{C})$ is

$$T = 4([\mathrm{Tr}(AB) - \alpha\beta]^2 - 4\,\mathrm{Tr}(A^\sharp B^\sharp) + 4\alpha \det A + 4\beta \det B), \tag{17}$$

and the tripartite entanglement measure is $\mathcal{T}_{123} = |T|$. Since under the action of $GL(6, \mathbb{C})$ this quantity takes up a nonzero factor, one immediately concludes that there must be at least two SLOCC equivalence classes of three-fermion states. In fact, by introducing the dual state and utilizing Plücker's relations one can complete the classification, and it turns out that we have four SLOCC orbits: the separable one, the biseparable one and two different types of true tripartite entanglement [13].

In order to be a true entanglement measure, this SLOCC invariant must also have the property that it does not increase under LOCC. Unfortunately, we do not have a proof for that, but numerical tests make us conjecture that this is the case.

3.1. Three qubits

This classification resembles that of the three-qubit system [1, 31], where the well-known representatives of the inequivalent classes with tripartite entanglement are the W and GHZ states. One may suspect that there is a connection between the two, and this indeed is the case. By looking at special three-fermion states one may observe that the space of three-qubit states, $\bigotimes_{i=1}^3 \mathbb{C}^2$, can be injected into our three-fermion one in such a way that the three-tangle [34] defined by Cayley's hyperdeterminant can be viewed as a special case of the quartic above.

To this end, we keep only the amplitudes with three different numbers in the subscript forgetting the overbars for a moment. We have eight such coefficients which is the number of coefficients needed to describe a three-qubit state. The natural way to do this is to choose an orthonormal basis, $\{f_0, f_1\} \in \mathbb{C}^2$, and take three-fold tensor products of them (computational basis). Now let us map an element, $f_i \otimes f_j \otimes f_k$, of this basis to $e_{1+3i} \wedge e_{2+3j} \wedge e_{3+3k} \in \bigwedge^3 \mathbb{C}^6$, i.e. the overbar indicates 1 and the lack of it indicates 0 on the place indexed by the number in the subscript. To a three-qubit state,

$$a = \sum_{i,j,k \in \{0,1\}} a_{ijk} f_i \otimes f_j \otimes f_k, \tag{18}$$

we associate this way an element $x = (\alpha, \beta, A, B)$ of $\mathfrak{M}$ where

$$\alpha = a_{000} \quad \beta = a_{111} \quad A = \begin{bmatrix} a_{011} & 0 & 0 \\ 0 & a_{101} & 0 \\ 0 & 0 & a_{110} \end{bmatrix} \quad B = \begin{bmatrix} a_{100} & 0 & 0 \\ 0 & a_{010} & 0 \\ 0 & 0 & a_{001} \end{bmatrix}. \tag{19}$$

For this element $\mathcal{T}_{123}$ equals the three tangle of a (using decimal notation):

$$T = 4((a_0a_7)^2 + (a_1a_6)^2 + (a_2a_5)^2 + (a_3a_4)^2) - 8(a_0a_7a_1a_6 + a_0a_7a_2a_5 + a_0a_7a_3a_4 + a_1a_6a_2a_5 + a_1a_6a_3a_4 + a_2a_5a_3a_4) + 16(a_0a_3a_5a_6 + a_7a_4a_2a_1). \tag{20}$$

In this way, the three-qubit system can be embedded in the three-fermion one.

An other way to look at the similarity between the two systems is obtained by observing that starting with the cubic Jordan algebra, $J_{1+1+1} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, the Freudenthal construction leads to the $V_2 \otimes V_2 \otimes V_2$ representation of the group $SL(2, \mathbb{C})^3$, and the quartic polynomial preserved by the action of the group is Cayley's hyperdeterminant. In section 2, we have seen that J_{1+1+1} is isomorphic to the subalgebra of J_3 of diagonal matrices.

For an element $x = (x_1, x_2, x_3) \in J_{1+1+1}$, we have a cubic norm $N(x) = x_1x_2x_3$, the sharp map assigning $x^\sharp = (x_2x_3, x_1x_3, x_1x_2)$ to x and on J we have a bilinear form whose value is $(x, y) = x_1y_1 + x_2y_2 + x_3y_3$ for $y = (y_1, y_2, y_3)$.

We see that the injection of the space of three-qubit states into the space of three-fermion states can be done at the Jordan algebra level. Moreover, the quartic invariant is based entirely on the cubic Jordan algebra structure in both cases. It is not surprising, therefore, that the three-tangle of a three-qubit state can also be obtained by first taking the associated special three-fermion state and then calculating the value of the quartic invariant on it.

3.2. One distinguished qubit with two bosonic qubits

According to the literature on Freudenthal triple systems [16], there are three more Jordan algebras for which the Freudenthal construction yields a representation that has a natural interpretation in quantum information theory. These are $J_1 = \mathbb{C}$, $J_{1+1} = \mathbb{C} \oplus \mathbb{C}$ and $J_{1+2} = \mathbb{C} \oplus M_2(\mathbb{C})$. We have seen in section 2 that these are all isomorphic to subalgebras of J_3.

The first two cases correspond to three bosonic qubits and a composite system consisting of one qubit and two other indistinguishable bosonic qubits. The Hilbert space of these systems can be naturally viewed as subspaces of that describing three qubits, so one might expect that these can be embedded into the latter much like the three-qubit system is embedded in the three-fermion one. The last one corresponds to a system containing a qubit and two indistinguishable fermionic particles with four single-particle states.

First take a look at J_{1+1}. With the Freudenthal construction we obtain a representation of $SL(2, \mathbb{C})^2$ on $\mathbb{C} \oplus \mathbb{C} \oplus J_{1+1} \oplus J_{1+1}$ that is isomorphic to $V_2 \otimes \mathrm{Sym}^2 V_2$. This enables us to classify entangled states in the space of a distinguishable and two bosonic qubits.

Let $\{e_0, e_1\}$ be the computational basis of $\mathbb{C}^2$, and let $f_0 = e_0 \otimes e_0$, $f_1 = e_0 \otimes e_1 + e_1 \otimes e_0$ and $f_2 = e_1 \otimes e_1$. Now a normalized vector in $\mathbb{C}^2 \otimes \mathrm{Sym}^2\mathbb{C}^2$ may be written as

$$b = \sum_{i=0}^{1}\sum_{j=0}^{2} b_{ij} e_i \otimes f_j, \tag{21}$$

where

$$\sum_{i=0}^{1}(|b_{i0}|^2 + 2|b_{i1}|^2 + |b_{i2}|^2) = 1. \tag{22}$$

The corresponding three-fermion state is given by $x = (\alpha, \beta, A, B) \in \mathfrak{M}$, where

$$\alpha = b_{00} \quad \beta = b_{12} \quad A = \begin{bmatrix} b_{02} & 0 & 0 \\ 0 & b_{11} & 0 \\ 0 & 0 & b_{11} \end{bmatrix} \quad B = \begin{bmatrix} b_{10} & 0 & 0 \\ 0 & b_{01} & 0 \\ 0 & 0 & b_{01} \end{bmatrix}. \tag{23}$$

For the state b we have the quartic invariant:

$$T = 4\big(b_{00}^2 b_{12}^2 + b_{02}^2 b_{10}^2\big) + 16\big(b_{11}^2 b_{00} b_{02} + b_{01}^2 b_{10} b_{12}\big) \\ - 8 b_{00} b_{02} b_{10} b_{12} - 16(b_{01} b_{02} b_{10} b_{11} + b_{00} b_{01} b_{11} b_{12}). \tag{24}$$

We are not aware of any application of the quartic invariant above within quantum information theory. However, it is interesting to note that our invariant appears within the realm of black hole solutions in string theory and quantum gravity. The model in question is the so-called st^2 model [35] which can be regarded as one coming from the stu-model [36, 37] after a $t = u$ degeneracy. In this model, the black hole entropy is expressed in terms of six charges (three magnetic and three electric) which can be mapped bijectively to the six amplitudes of our state b. This correspondence between an entanglement measure on one side and the black hole entropy formula for a particular black hole solution on the other forms the basis of the black hole analogy, our guiding principle in constructing this measure.

3.3. Three bosonic qubits

Now let us turn to the Jordan algebra $J_1 = \mathbb{C}$ in which the norm of an element is simply the cube of it, the sharp means taking the square, and the trace bilinear form of two elements $x, y \in J_1$ is $3xy$. Again after some calculation one can show that J_1 is a subalgebra of J_{1+1}, the inclusion map being $x \mapsto (x, x)$. The Freudenthal construction in this case leads to a four-dimensional representation of $SL(2, \mathbb{C})$ isomorphic to $\mathrm{Sym}^3 V_2$ which is related to the system of three indistinguishable bosonic qubits.

A general normalized state in $\mathrm{Sym}^3\mathbb{C}^2$ may be written as

$$c = c_0 e_0 \otimes e_0 \otimes e_0 + c_3 e_1 \otimes e_1 \otimes e_1 + c_1(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1) \\ + c_2(e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0), \tag{25}$$

where $|c_0|^2 + |c_3|^2 + 3(|c_1|^2 + |c_2|^2) = 1$. To this we associate the element $x = (c_0, c_3, c_2 I_3, c_1 I_3)$ in $\mathfrak{M}$, where I_n denotes the identity element of $M_n(\mathbb{C})$. For this state the quartic invariant is

$$T = 4c_0^2 c_3^2 - 12 c_1^2 c_2^2 - 24 c_0 c_1 c_2 c_3 + 16\big(c_0 c_2^3 + c_3 c_1^3\big). \tag{26}$$

Our entanglement measure for three bosonic qubits has appeared in the context of stringy black holes as the black hole entropy formula in the so-called t^3-model [35, 38]. For the interesting geometry of three bosonic qubits see the paper of Brody *et al* [39].

3.4. One qubit and two fermions with four single particle states

The remaining Jordan algebra is $J_{1+2} = \mathbb{C} \oplus M_2(\mathbb{C})$. Let $x = (\alpha, x_0)$ and $y = (\beta, y_0)$ be two elements of J_{1+2}. The cubic norm form is given by $N(x) = \alpha \det x_0$, the sharp map is $x^\sharp = (\det x_0, \alpha(\mathrm{Tr}\, x_0) I_2 - \alpha x_0)$, and finally the trace bilinear map in this case is $(x, y) \mapsto \alpha\beta + \mathrm{Tr}(x_0 y_0)$. Once again one can check that this Jordan algebra can naturally be viewed as a subalgebra of J_3, namely it is isomorphic to the subalgebra of block-diagonal matrices with a 1×1 and a 2×2 block in the diagonal.

We know that the Freudenthal construction applied to J_{1+2} yields a representation of $SL(2, \mathbb{C}) \times SL(4, \mathbb{C})$ isomorphic to $V_2 \otimes \bigwedge^2 V_4$. Moreover, it is known that the fermionic measure describing the bipartite entanglement of two fermions with four single-particle states reduces to the two-qubit concurrence [11, 13] in the same way as the three-fermion measure reduces to the three-tangle; hence, one may expect that in a sense this system fits between the three-fermion and the three-qubit one. This expectation is further supported by the fact that we have the embeddings $J_{1+1+1} \subset J_{1+2} \subset J_3$.

Table 1. Subspaces of $\mathfrak{M}$ associated with Hilbert spaces $\mathcal{H} \subset \mathcal{H}_0$ describing various quantum-mechanical systems with the SLOCC group $G \subset G_0$.

$\mathcal{H}$	G	Remark
$\mathcal{H}_0 = \bigwedge^3 \mathbb{C}^6$	$G_0 = GL(6, \mathbb{C})$	
$\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4$	$GL(2, \mathbb{C}) \times GL(4, \mathbb{C})$	$A, B \in M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	$GL(2, \mathbb{C})^3$	A, B diagonal
$\mathbb{C}^2 \otimes \mathrm{Sym}^2\mathbb{C}^2$	$GL(2, \mathbb{C})^2$	$A, B \in M_1(\mathbb{C}) \oplus \mathbb{C} \cdot I_2$
$\mathrm{Sym}^3\mathbb{C}^2$	$GL(2, \mathbb{C})$	$A, B \in \mathbb{C} \cdot I_3$

Let us see how this works explicitly. Let $\{e_0, e_1\}$ and $\{f_0, f_1, f_2, f_3\}$ be the canonical basis of $\mathbb{C}^2$ and $\mathbb{C}^4$, respectively. A state, $d \in \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4$, may be written as

$$d = \sum_{i=0}^{1} \sum_{0 \leqslant j < k \leqslant 3} d_{ijk} e_i \otimes (f_j \wedge f_k), \tag{27}$$

the amplitudes being antisymmetric in the second and the third index. The condition of being normalized means that

$$\sum_{i=0}^{1} \sum_{0 \leqslant j < k \leqslant 3} |d_{ijk}|^2 = 1. \tag{28}$$

Now we relate the six-state labels to the two-state and four-state ones as $(1, \bar{1}) \mapsto (0, 1)$ and $(2, 3, \bar{2}, \bar{3}) \mapsto (0, 1, 2, 3)$. respectively, and keep only the 12 coefficients whose index contains precisely one of 1 and $\bar{1}$. We associate with d the element $x = (\alpha, \beta, A, B) \in \mathfrak{M}$, where

$$\alpha = d_{001} \quad \beta = d_{123} \quad A = \begin{bmatrix} d_{023} & 0 & 0 \\ 0 & d_{103} & d_{120} \\ 0 & d_{113} & d_{121} \end{bmatrix} \quad B = \begin{bmatrix} d_{101} & 0 & 0 \\ 0 & d_{021} & d_{002} \\ 0 & d_{031} & d_{003} \end{bmatrix}. \tag{29}$$

For this state the value of the quartic tripartite entanglement measure is

$$\begin{aligned} T = {} & 4((d_{023}d_{101})^2 + (d_{021}d_{103})^2 + (d_{002}d_{113})^2 + (d_{031}d_{120})^2 + (d_{003}d_{121})^2 + (d_{001}d_{123})^2) \\ & + 8(d_{002}d_{021}d_{103}d_{113} + d_{021}d_{031}d_{103}d_{120} + d_{002}d_{003}d_{113}d_{121} + d_{003}d_{031}d_{120}d_{121}) \\ & + 16(d_{003}d_{021}d_{113}d_{120} + d_{001}d_{023}d_{103}d_{121} + d_{002}d_{031}d_{103}d_{121} + d_{003}d_{021}d_{101}d_{123}) \\ & - 16(d_{001}d_{023}d_{113}d_{120} + d_{002}d_{031}d_{101}d_{123}) - 8(d_{021}d_{023}d_{101}d_{103} \\ & + d_{002}d_{023}d_{101}d_{113} + d_{023}d_{031}d_{101}d_{120} + d_{002}d_{031}d_{113}d_{120} + d_{003}d_{023}d_{101}d_{121} \\ & + d_{003}d_{021}d_{103}d_{121} + d_{001}d_{023}d_{101}d_{123} + d_{001}d_{021}d_{103}d_{123} \\ & + d_{001}d_{002}d_{113}d_{123} + d_{001}d_{031}d_{120}d_{123} + d_{001}d_{003}d_{121}d_{123}). \end{aligned} \tag{30}$$

3.5. Representatives of SLOCC equivalence classes

To sum up, we have the chain of embeddings of Jordan algebras, $J_1 \subset J_{1+1} \subset J_{1+1+1} \subset J_{1+2} \subset J_3$, that gives rise via Freudenthal's construction to the chain of embeddings of Hilbert spaces, $\mathrm{Sym}^3\mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathrm{Sym}^2\mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \subset \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4 \subset \bigwedge^2 \mathbb{C}^6$. The appropriate subspaces of $\mathfrak{M}$ are shown in table 1 along with their SLOCC group. These embeddings are compatible with the SLOCC classification of entanglement in the sense that SLOCC orbits of any of these systems are subsets of the intersections of SLOCC orbits of the three-fermion Hilbert space with the appropriate subspace. In order to find representatives of various entanglement classes, it is therefore enough to look for them in the smallest possible subspace then interpret it as elements of the larger Hilbert spaces. These representatives can be chosen to be the following ones:

$$\mathrm{GHZ} = \frac{1}{\sqrt{2}}(1, 1, 0, 0), \tag{31}$$

$$W = \frac{1}{\sqrt{3}}(0, 0, 0, I_3), \tag{32}$$

$$B_1 = \frac{1}{\sqrt{2}}\left(1, 0, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0\right), \tag{33}$$

$$B_2 = \frac{1}{\sqrt{2}}\left(1, 0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0\right), \tag{34}$$

$$B_3 = \frac{1}{\sqrt{2}}\left(1, 0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0\right), \tag{35}$$

$$S = (1, 0, 0, 0). \tag{36}$$

The GHZ and W states show tripartite entanglement; B_i is a biseparable and S is a separable state. Apart from B_i these can be found in the system of three bosonic qubits, but the relations characterizing states of rank at most two imply separability in this case. Therefore, the representative of the biseparable class is chosen from the larger Hilbert space, $\mathbb{C}^2 \otimes \mathrm{Sym}^2\mathbb{C}^2$. Of the biseparable subclasses only B_1 is present in the latter, all can be found in the three-qubit case, B_2 and B_3 are equivalent in $\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4$, and all three are equivalent in the largest Hilbert space $\bigwedge^3 \mathbb{C}^6$. Table 2 shows these states for each system.

4. Embedding systems with distinguishable constituents into fermionic ones

Looking at these results one might ask what parts of this process can be generalized. Since the systems considered above are the only ones which are related to Freudenthal triples we cannot extend these results for more general quantum systems. On the other hand, the phenomenon that the Hilbert space of a smaller system is embedded in the space of a larger one appears in more general cases.

To be more specific, suppose that we have the invariants $I_1, \ldots, I_n$ under the SLOCC group G represented on a Hilbert space $\mathcal{H}$. It may happen that there exists a subgroup $G' \subset G$ that can be viewed as a SLOCC group of a quantum-mechanical system to which we can associate a subspace $\mathcal{H}' \subset \mathcal{H}$ of our original Hilbert space in such a way that $\mathcal{H}'$ is invariant under the action of G'. Obviously, the invariants $I_1, \ldots, I_n$ restricted to $\mathcal{H}'$ are invariant under the subgroup G' of SLOCC transformations of the smaller system. This means that if we have two states in $\mathcal{H}'$ that are G-inequivalent, then they will necessarily be G'-inequivalent. However, it may also happen that two G-equivalent states cannot be transformed to each other by the smaller SLOCC group G'; in other words, the intersection of a SLOCC orbit in $\mathcal{H}$ with $\mathcal{H}'$ regarded as a set in $\mathcal{H}'$ may split to several disjoint orbits under the smaller SLOCC group G'. In this case, further refinement is needed to obtain full classification of the entangled states in $\mathcal{H}'$.

Table 2. Representatives of SLOCC orbits of quantum-mechanical systems classified via Freudenthal's construction.

Space ($\mathcal{H}$)	Representatives
$\bigwedge^3 \mathbb{C}^6$	$\mathrm{GHZ} = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6)$
	$W = \frac{1}{\sqrt{3}}(e_4 \wedge e_2 \wedge e_3 + e_1 \wedge e_5 \wedge e_3 + e_1 \wedge e_2 \wedge e_6)$
	$B_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_5 \wedge e_6)$
	$S = e_1 \wedge e_2 \wedge e_3$
$\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4$	$\mathrm{GHZ} = \frac{1}{\sqrt{2}}(e_0 \otimes (f_0 \wedge f_1) + e_1 \otimes (f_2 \wedge f_3))$
	$W = \frac{1}{\sqrt{3}}(e_0 \otimes (f_2 \wedge f_3) + e_1 \otimes (f_0 \wedge f_3) + e_1 \otimes (f_2 \wedge f_1))$
	$B_1 = \frac{1}{\sqrt{2}} e_0 \otimes (f_0 \wedge f_1 + f_2 \wedge f_3)$
	$B_2 = \frac{1}{\sqrt{2}}(e_0 \otimes (f_0 \wedge f_1) + e_1 \otimes (f_0 \wedge f_3))$
	$S = e_0 \otimes (f_0 \wedge f_1)$
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	$\mathrm{GHZ} = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)$
	$W = \frac{1}{\sqrt{3}}(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1)$
	$B_1 = \frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0 + e_1 \otimes e_1))$
	$B_2 = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1)$
	$B_3 = \frac{1}{\sqrt{2}}((e_0 \otimes e_0 + e_1 \otimes e_1) \otimes e_0)$
	$S = e_0 \otimes e_0 \otimes e_0$
$\mathbb{C}^2 \otimes \mathrm{Sym}^2\mathbb{C}^2$	$\mathrm{GHZ} = \frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0) + e_1 \otimes (e_1 \otimes e_1))$
	$W = \frac{1}{\sqrt{3}}(e_1 \otimes (e_0 \otimes e_0) + e_0 \otimes (e_1 \otimes e_0 + e_0 \otimes e_1))$
	$B_1 = \frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0 + e_1 \otimes e_1))$
	$S = e_0 \otimes (e_0 \otimes e_0)$
$\mathrm{Sym}^3\mathbb{C}^2$	$\mathrm{GHZ} = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)$
	$W = \frac{1}{\sqrt{3}}(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1)$
	$S = e_0 \otimes e_0 \otimes e_0$

4.1. Separability in arbitrary systems

As an example, suppose that we have N types of fermionic particles, k_i of the ith type having n_i single-particle states. To this composite system we associate the Hilbert space $\mathcal{H} = \bigwedge^{k_1} \mathcal{H}_1^{(0)} \otimes \cdots \otimes \bigwedge^{k_N} \mathcal{H}_N^{(0)}$, where $\dim \mathcal{H}_i^{(0)} = n_i$ and the SLOCC group $G = GL(n_1, \mathbb{C}) \times \cdots \times GL(n_N, \mathbb{C})$. This space can be embedded in $\mathcal{K} = \bigwedge^{k_1+\cdots+k_N} \left(\mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)}\right)$ via the linear map $\varphi : \mathcal{H} \to \mathcal{K}$ defined by

$$\begin{aligned}\varphi : (v_1 \wedge \ldots \wedge v_{k_1}) \otimes (v_{k_1+1} \wedge \ldots \wedge v_{k_1+k_2}) \otimes \cdots \otimes (v_{k_1+\cdots+k_{N-1}+1} \wedge \ldots \wedge v_{k_1+\cdots+k_N}) \\ \mapsto v_1 \wedge v_2 \wedge \ldots \wedge v_{k_1+\cdots+k_N}\end{aligned} \tag{37}$$

on tensor products of decomposable vectors and its SLOCC group $H = GL(n_1 + \cdots + n_N, \mathbb{C})$ contains G in an obvious way. The question that arises naturally is the following: what is the connection between the set of unentangled states in the embedded and the embedding space? The answer is given by the following theorem whose proof can be found in the appendix.

Theorem. *Let $N \in \mathbb{N}$, $(k_i)_{i=1}^N$ and $(n_i)_{i=1}^N$ be N-tuples of positive integers, $\mathcal{H}_i^{(0)}$ a Hilbert space of dimension n_i, $\mathcal{H}_i = \bigwedge^{k_i} \mathcal{H}_i^{(0)}$ (for all $i \in \{1, \ldots, N\}$), $\mathcal{K}^{(0)} = \bigoplus_{i=1}^N \mathcal{H}_i^{(0)}$ and*

$\mathcal{K} = \bigwedge^k \mathcal{K}^{(0)}$, where $k = \sum_{i=1}^N k_i$. Let $\varphi : \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \to \mathcal{K}$ be the linear map defined by $(e_{1,j_1} \wedge \ldots \wedge e_{1,j_{k_1}}) \otimes \cdots \otimes (e_{N,j_{k-k_N+1}} \wedge \ldots \wedge e_{N,j_k}) \mapsto e_{1,j_1} \wedge \ldots \wedge e_{1,j_{k_1}} \wedge \ldots \wedge e_{N,j_{k-k_N+1}} \wedge \ldots \wedge e_{N,j_k}$ for some orthonormal bases $\{e_{i,j}\}_{j=1}^{k_i} \subset \mathcal{H}_i^{(0)}$. Then a vector v in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N = \mathcal{H}$ is a tensor product of decomposable vectors in $\mathcal{H}_i$ iff $\varphi(v) \in \mathcal{K}$ is decomposable.

This theorem applied to our scenario means that a state ψ in $\mathcal{H}$ is separable (tensor product of decomposable states) iff $\varphi(\psi) \in \mathcal{K}$ is separable (decomposable).

4.2. Plücker relations

Since a state in $\mathcal{K}$ is separable precisely when the Plücker relations hold [13, 40, 41], we can conclude that the Plücker relations provide a sufficient and necessary condition of separability of an arbitrary system of finitely many particles.

Recall that the Plücker relations say that a k fermionic state with totally antisymmetric amplitudes $P_{j_1 j_2 \ldots j_k}$ is separable iff for any $\mathcal{A} = \{a_1, \ldots, a_{k-1}\} \subset I$ and $\mathcal{B} = \{b_1, \ldots, b_{k+1}\} \subset I$ the polynomial,

$$\Pi_{\mathcal{A},\mathcal{B}}(P) = \sum_{j=1}^{k+1} (-1)^{j-1} P_{a_1 \ldots a_{k-1} b_j} P_{b_1 \ldots b_{j-1} b_{j+1} \ldots b_{k+1}}, \tag{38}$$

equals to zero. However, now our fermionic state is of special kind namely it is of the form

$$\varphi(\psi) = \sum_{J \subset I} P_{j_1 j_2 \ldots j_k} e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k}, \tag{39}$$

where $J = \{j_1, \ldots, j_k\}$ and $I = \bigcup_{i=1}^N I_i = \{1, \ldots, n\}$ with $I_1 = \{1, \ldots, n_1\}, I_2 = \{n_1+1, \ldots, n_1+n_2\}, \ldots, I_N = \{n-n_N+1, \ldots, n\}$ ($n = \sum_{i=1}^N n_i$ and $\{e_j\}_{j \in I_i}$ is an orthonormal basis of $\mathcal{H}_i^{(0)}$). Hence we do not need to consider all relations since many of them are identically zero due to the very special form of the vectors in ran φ. This means that $P_{j_1 j_2 \ldots j_k} = 0$ if there exists $i \in \{1, \ldots, N\}$ such that $|J \cap I_i| \neq k_i$. Using this property of the coefficients $P_{j_1 \ldots j_k}$ one can see that $\Pi_{\mathcal{A},\mathcal{B}}(P)$ is identically zero unless

$$(\exists j \in \{1, \ldots, k+1\})(\forall i \in \{1, \ldots, N\})(|(\mathcal{A} \cup \{b_j\}) \cap I_i| = |(\mathcal{B} \setminus \{b_j\}) \cap I_i| = k_i) \tag{40}$$

which is equivalent to

$$\begin{aligned}(\exists j \in \{1, \ldots, k+1\})(\forall\, i \in \{1, \ldots, N\})((b_j \notin I_i) \quad &\text{and} \quad (|\mathcal{A} \cap I_i| = |\mathcal{B} \cap I_i| = k_i)) \\ \text{or} \quad ((b_j \in I_i) \quad &\text{and} \quad (|\mathcal{A} \cap I_i| + 1 = |\mathcal{B} \cap I_i| - 1 = k_i))).\end{aligned} \tag{41}$$

When this holds for some j we have exactly $|(\mathcal{B} \setminus \mathcal{A}) \cap I_i|$ nonvanishing terms where i is the unique index for which $b_j \in I_i$.

As a special case, take the system of N qubits described by the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. With the notations above this corresponds to $k_1 = k_2 = \cdots = k_N = 1$ and $n_1 = n_2 = \cdots = n_N = 2$; therefore, we can embed $\mathcal{H}$ in $\mathcal{K} = \bigwedge^N \mathbb{C}^{2N}$. It is not too hard to check that in this case the Plücker relations tell us that an element,

$$\psi = \sum_{i_1, \ldots, i_N = 0}^{1} \psi_{i_1 \ldots i_N} e_{i_1} \otimes \cdots \otimes e_{i_N}, \tag{42}$$

is separable if and only if for all $1 \leqslant j \leqslant N$ and for all $(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_N, h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_N) \in \mathbb{Z}_2^{2N-2}$ we have

$$\psi_{i_1 \ldots i_{j-1} 0 i_{j+1} \ldots i_N} \psi_{h_1 \ldots h_{j-1} 1 h_{j+1} \ldots h_N} = \psi_{i_1 \ldots i_{j-1} 1 i_{j+1} \ldots i_N} \psi_{h_1 \ldots h_{j-1} 0 h_{j+1} \ldots h_N} \tag{43}$$

which is indeed the separability condition for N qubits. Note that for $N = 2$ (two qubits) we obtain the result of Gittings and Fischer [11] of relating the concurrence to the fermionic measure of Schliemann [9] for a *special* two-fermion state with merely four nontrivial amplitudes.

4.3. Entangled states

This scheme of embedding an arbitrary system of finitely many particles has the property that separability in the larger space implies separability in the smaller one. Since separable states are SLOCC equivalent to each other, we conclude that this special SLOCC equivalence class does not split into subclasses when we restrict ourselves to the smaller space and its smaller SLOCC group. Note, however, that this by no means is the case with other SLOCC classes. For example, look at the system of four qubits that can be embedded in $\mathcal{K} = \bigwedge^4 \mathbb{C}^8$. Let $\{e_{2i-1}, e_{2i}\}$ be the computational basis in the Hilbert space of the ith qubit. Then the states,

$$P = \frac{1}{\sqrt{2}} e_1 \otimes e_3 \otimes (e_5 \otimes e_7 + e_6 \otimes e_8) \tag{44}$$

$$Q = \frac{1}{\sqrt{2}} (e_1 \otimes e_3 + e_2 \otimes e_4) \otimes e_5 \otimes e_7, \tag{45}$$

cannot be transformed into each other since P is $AB(CD)$-separable but Q is $(AB)CD$-separable. But their images in $\mathcal{K}$ are (for the definition of φ see equation (23))

$$\varphi(P) = \frac{1}{\sqrt{2}} (e_1 \wedge e_3 \wedge e_5 \wedge e_7 + e_1 \wedge e_3 \wedge e_6 \wedge e_8), \tag{46}$$

$$\varphi(Q) = \frac{1}{\sqrt{2}} (e_1 \wedge e_3 \wedge e_5 \wedge e_7 + e_2 \wedge e_4 \wedge e_5 \wedge e_7), \tag{47}$$

which can be transformed to each other by applying the element $X \otimes I \otimes I \in GL(8, \mathbb{C})$; therefore, they are SLOCC-equivalent in $\mathcal{K}$. (Here $X = \sigma_1$, i.e. the first Pauli matrix.)

This example convinces us that the phenomenon of splitting of SLOCC classes is not uncommon. In fact, it does appear even in the three-qubit case where $(AB)C$-, $(BC)A$- and $(CA)B$-biseparable states are inequivalent but their images in the three-fermion system belong to the same class. However, this embedding is rather special because there is 'no room' for this kind of splitting of multipartite entangled states, and this may be a main reason of the similarity of the classification of entanglement in the two systems.

Despite the fact that the entanglement measures in the embedding system may be much coarser than needed for full SLOCC classification of the smaller system, this method might prove to be a useful tool. If the splitting of SLOCC classes could be fully understood then it would be enough to identify the entanglement classes of $\bigwedge^k \mathbb{C}$ which might have a simpler structure than the Hilbert space of a general system containing various types of particles.

Let us now introduce a family of SLOCC-invariants for fermionic systems. Let $\mathcal{K}^{(0)} = \mathbb{C}^{dk}$ and $\mathcal{K} = \bigwedge^k \mathcal{K}^{(0)}$ where $k \in 2\mathbb{N}$ and $d \in \mathbb{N}$. Given a state $P \in \mathcal{K}$ we can take the d-fold wedge product of it which lives in $\bigwedge^{dk} \mathbb{C}^{dk}$. Hence it is invariant under the action of $SL(dk, \mathbb{C})$, and picks up a factor corresponding to the determinant under the action of $GL(dk, \mathbb{C})$. Let $\xi(P)$ denote the absolute value of this vector:

$$\|P \wedge P \wedge \ldots \wedge P\| = \xi(P). \tag{48}$$

Let $P_{i_1 \ldots i_k}$ denote the coefficients of P with respect to the induced basis:

$$P = \sum_{i_1 \ldots i_k = 1}^{dk} P_{i_1 \ldots i_k} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}. \tag{49}$$

After expanding the product above we obtain

$$P \wedge P \wedge \ldots \wedge P = \sum_{i_1,\ldots,i_{dk}=1}^{dk} P_{i_1\ldots i_k} P_{i_{k+1}\ldots i_{2k}} \ldots P_{i_{(d-1)k+1}\ldots i_{dk}} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{dk}}$$
$$= \sum_{\pi\in S_{dk}} \sigma(\pi) P_{\pi(1)\ldots\pi(k)} \ldots P_{\pi(dk-k+1)\ldots\pi(dk)} e_1 \wedge e_2 \wedge \ldots \wedge e_{dk}, \tag{50}$$

where S_n is the group of bijections form the set $\{1, \ldots, n\}$ to itself, and $\sigma : S_n \to \{1, -1\}$ is the alternating representation of this group. From this one can see that

$$\xi(P) = \left| \sum_{\pi\in S_{dk}} \sigma(\pi) P_{\pi(1)\ldots\pi(k)} P_{\pi(k+1)\ldots\pi(2k)} \ldots P_{\pi(dk-k+1)\ldots\pi(dk)} \right|. \tag{51}$$

Since $\xi((A \otimes \cdots \otimes A)P) = |\det A|\xi(P)$, it follows that if $P' = (A \otimes \cdots \otimes A)P$ then either both of $\xi(P)$ and $\xi(P')$ are 0 or none of them. Moreover, if $|\det A| = 1$ then $\xi(P) = \xi(P')$.

Now for $N \in 2\mathbb{N}$ and $d \in \mathbb{N}$ we can embed the Hilbert space, $\mathcal{H} = \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$, of N qudits into $\mathcal{K}$, the image of

$$\psi = \sum_{i_1,\ldots,i_N=1}^{d} \psi_{i_1 i_2\ldots i_N} e_{i_1} \otimes \cdots \otimes e_{i_N} \tag{52}$$

being

$$\tilde{\varphi}(\psi) = P = \sum_{i_1,\ldots,i_N=1}^{d} \psi_{i_1 i_2\ldots i_N} e_{i_1} \wedge e_{d+i_2} \wedge \ldots \wedge e_{(N-1)d+i_N}. \tag{53}$$

Note that the definition of $\tilde{\varphi}$ is slightly different from that of the map φ defined above, but the difference is only a relabelling of basis elements in $\mathcal{H}$. For this state the value of $\xi(P)$ is

$$\xi(P) = \left| \sum_{\pi_1,\ldots,\pi_N\in S_d} \left(\prod_{i=1}^{N} \sigma(\pi_i) \right) \psi_{\pi_1(1)\pi_2(1)\ldots\pi_N(1)} \psi_{\pi_1(2)\pi_2(2)\ldots\pi_N(2)} \ldots \psi_{\pi_1(d)\pi_2(d)\ldots\pi_N(d)} \right|. \tag{54}$$

With a slight abuse of notation $\xi(\psi)$ will denote $\xi(\tilde{\varphi}(\psi))$. Note that for the special case of an *even* number of qubits i.e. $d = 2$ the squared magnitude of the measure of equation (42) is related to that of Wong and Christensen [42]. For four qubits it is known that this measure boils down to that denoted by the letter H in the paper of Luque and Thibon [43].

As an example, take the following two globally entangled states in $\mathcal{H}$:

$$\psi = \frac{1}{\sqrt{d}}(e_1 \otimes \cdots \otimes e_1 + e_2 \otimes \cdots \otimes e_2 + \cdots + e_d \otimes \cdots \otimes e_d), \tag{55}$$

$$\phi = \frac{1}{\sqrt{N}}(e_2 \otimes e_1 \otimes \cdots \otimes e_1 + e_1 \otimes e_2 \otimes e_1 \otimes \cdots \otimes e_1 + \cdots + e_1 \otimes \cdots \otimes e_1 \otimes e_2). \tag{56}$$

Then we have

$$\xi(\psi) = \sum_{\pi\in S_d} \left(\frac{1}{\sqrt{d}}\right)^d = d!d^{-\frac{d}{2}} \neq 0 \tag{57}$$

but $\xi(\phi) = 0$ for $N > 2$; hence we can conclude that ψ and ϕ are not SLOCC equivalent.

4.4. Reduced density matrices

In many cases one can gain information about a fermionic system by looking at its single-particle reduced density matrix. The mapping described above takes a pure state of an arbitrary system and maps it to a special fermionic one; therefore, the question naturally arises: how are the one particle reduced density matrices of the two states related to each other?

Let $\mathcal{H}_i = \bigwedge^{k_i} \mathcal{H}_i^{(0)}, \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N, \mathcal{K}^{(0)} = \mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)}$ and $\mathcal{K} = \bigwedge^k \mathcal{K}^{(0)}$ as before ($k = k_1 + \cdots + k_N$), and let $\{e_a\}_{a\in I_i}$ be orthonormal bases in $\mathcal{H}_i^{(0)}$, respectively where $\dim \mathcal{H}_i^{(0)} = |I_i| = n_i$ $(i \in \{1, \ldots, N\})$, $n := n_1 + \cdots + n_N$. The function $\varphi : \mathcal{H} \to \mathcal{K}; (v_1 \wedge \ldots \wedge v_{k_1}) \otimes \cdots \otimes (v_{k-k_N+1} \wedge \ldots \wedge v_k) \mapsto v_1 \wedge \ldots \wedge v_k$ maps a general state,

$$\psi = \sum_{\substack{i_1,\ldots,\\ i_{k_1}\in I_1}} \sum_{\substack{i_{k_1+1},\ldots,\\ i_{k_1+k_2}\in I_2}} \cdots \sum_{\substack{i_{k-k_N+1},\\ \ldots,i_k\in I_N}} \psi_{i_1 i_2 \ldots i_k} (e_{i_1} \wedge \ldots \wedge e_{i_{k_1}}) \otimes \cdots \otimes (e_{i_{k-k_N+1}} \wedge \ldots \wedge e_{i_k}), \tag{58}$$

to

$$\varphi(\psi) = P = \sum_{\substack{i_1,\ldots,\\ i_{k_1}\in I_1}} \sum_{\substack{i_{k_1+1},\ldots,\\ i_{k_1+k_2}\in I_2}} \cdots \sum_{\substack{i_{k-k_N+1},\\ \ldots,i_k\in I_N}} \psi_{i_1 i_2 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}. \tag{59}$$

Let ρ denote the one particle density matrix of the state P and $\rho_i (i \in \{1, \ldots, N\})$:

$$\begin{aligned}
&\rho = \mathrm{Tr}_{2,3,\ldots,k}\, P P^* = (\mathrm{id}_{\mathcal{K}^{(0)}} \otimes \mathrm{Tr} \otimes \cdots \otimes \mathrm{Tr}) P P^* \in M(n, \mathbb{C})\\
&\rho_1 = \mathrm{Tr}_{2,3,\ldots,k}\, \psi\psi^* = (\mathrm{id}_{\mathcal{H}_1^{(0)}} \otimes \mathrm{Tr} \otimes \cdots \otimes \mathrm{Tr}) \psi\psi^* \in M(n_1, \mathbb{C})\\
&\vdots\\
&\rho_N = \mathrm{Tr}_{1,2,\ldots,k-k_N,k-k_N+2,\ldots,k}\, \psi\psi^* \in M(n_N, \mathbb{C}).
\end{aligned} \tag{60}$$

Using identities like

$$\begin{aligned}
&\mathrm{Tr}_{2,3,\ldots,k} \left(e_{i_1} \wedge \ldots \wedge e_{i_k}\right)\left(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*\right) = \frac{1}{k}\left(e_{i_1} e_{i_1}^* + \cdots + e_{i_k} e_{i_k}^*\right)\\
&\mathrm{Tr}_{2,3,\ldots,k} \left(e_{i_1} \wedge \ldots \wedge e_{i_k}\right)\left(e_{i_1'}^* \wedge \ldots \wedge e_{i_k}^*\right) = \frac{1}{k} e_{i_1} e_{i_1'}^*\\
&\mathrm{Tr}_{2,3,\ldots,k} \left[\left(e_{i_1} \wedge \ldots \wedge e_{k_1}\right) \otimes \cdots \otimes \left(e_{i_{k-k_N+1}} \wedge \ldots \wedge e_{i_k}\right)\right]\\
&\left[\left(e_{i_1}^* \wedge \ldots \wedge e_{k_1}^*\right) \otimes \cdots \otimes \left(e_{i_{k-k_N+1}}^* \wedge \ldots \wedge e_{i_k}^*\right)\right] = \frac{1}{k_1}\left(e_{i_1} e_{i_1}^* + \cdots + e_{i_{k_1}} e_{i_{k_1}}^*\right)\\
&\mathrm{Tr}_{2,3,\ldots,k} \left[\left(e_{i_1} \wedge \ldots \wedge e_{k_1}\right) \otimes \cdots \otimes \left(e_{i_{k-k_N+1}} \wedge \ldots \wedge e_{i_k}\right)\right]\\
&\left[\left(e_{i_1'}^* \wedge \ldots \wedge e_{k_1}^*\right) \otimes \cdots \otimes \left(e_{i_{k-k_N+1}}^* \wedge \ldots \wedge e_{i_k}^*\right)\right] = \frac{1}{k_1} e_{i_1} e_{i_1'}^*
\end{aligned} \tag{61}$$

we can see that

$$\rho = \bigoplus_{i=1}^{N} \frac{k_i}{k} \rho_i. \tag{62}$$

Since both sides are linear in the density matrices of the whole systems, this holds for mixed states too. An alternative normalization is given by $\gamma = k\rho$ and $\gamma_i = k_i \rho_i$, the relation for these is $\gamma = \bigoplus_{i=1}^{N} \gamma_i$ $(i \in \{1, \ldots, N\})$. Observe that the two states are unentangled iff $\gamma_i^2 = \gamma_i$ $(i \in \{1, \ldots, N\})$ and $\gamma^2 = \gamma$, respectively, yielding an alternative proof of our theorem.

4.5. Physical interpretation

So far we have regarded this relation of systems of distinguishable particles and fermionic ones as a purely mathematical one. However, we can interpret it as a physical property of the particles pretending they are *really* indistinguishable but for some reason they are not in the same state of some inner degree of freedom (analogous to isospin) and these inner states are not mixed by the Hamiltonian.

As the simplest example, suppose we have two distinguishable qubits so each particle has two states, call them e_0, e_1 and e'_0, e'_1. We can combine the four states of the two particles into a single Hilbert space having dimension 4. If the Hamiltonian governing the evolution of a state in this space has vanishing matrix elements between basis states with and without a prime then the subspaces spanned by $\{e_0, e_1\}$ and $\{e'_0, e'_1\}$ are not mixed, and therefore we can associate an inner quantum number with the states. A state which is a linear combination of e_0 and e_1 can be called a particle of type one and a state in the span of $\{e'_0, e'_1\}$ can be called a particle of type two. If we take two fermionic particles having these four single-particle states, and one of them is of type one and the other is of type two then they will behave exactly as if they were distinguishable qubits.

5. Conclusions

In this paper, we have studied quantum systems containing both distinguishable and identical constituents. A special subclass of such systems can be studied using the algebraic constructs called Freudenthal systems. The corresponding physical systems of this subclass are the *tripartite* ones that can be embedded to a system consisting of three fermions with six single-particle states. Such embedded systems are those consisting of a qubit and a bipartite fermionic system with four single-particle states, three ordinary qubits, three bosonic qubits and two bosonic qubits coupled to an ordinary qubit. For these systems we presented a *complete* classification of SLOCC orbits, based on the quartic SLOCC invariant arising from the corresponding one of the associated Freudenthal system. Though the full appreciation of these invariants within the field of quantum information is still missing, we have pointed out that they arise quite naturally as entropy formulae for black hole solutions in supergravity theories.

As a next step retaining merely the idea of embedding one type of system to the other we studied issues of separability for systems embedded into fermionic ones. We proved that the Plücker relation for these embedding fermionic systems play a universal role in checking the separability of the embedded ones. We briefly elaborated also on the interesting problem of splitting of SLOCC classes when comparing the entanglement properties of the embedding and embedded systems. Such considerations enabled the construction of a class of pure state entanglement measures containing some well-known ones as a limiting case. Since in many cases we can gain information about a fermionic system by looking at its single-particle reduced density matrices, a natural question to be addressed is that how these density matrices for the embedding and embedded systems are related? We have answered this question by presenting an explicit formula. And at last we proposed a possible physical interpretation of our embedding of one type of system to the other. The conclusion is that we can regard our embedding trick as a convenient representation for a quantum system with particles which are *really* indistinguishable but for some reason they are not in the same state of some *inner* degree of freedom.

Acknowledgments

Financial support from the Országos Tudományos Kutatási Alap (grant numbers T047035, T047041, T038191) is gratefully acknowledged.

Appendix. Proof of the mathematical results

Lemma. *Let $\mathcal{H}_A \simeq \mathbb{C}^{n_A}$ and $\mathcal{H}_B \simeq \mathbb{C}^{n_B}$ be two Hilbert spaces, and $\mathcal{H} = \bigwedge^{k_A+k_B}(\mathcal{H}_A \oplus \mathcal{H}_B)$ for some $k_A, k_B \in \mathbb{N}$. Suppose that $v \in \mathcal{H} \setminus \{0\}$ is decomposable and $v \in \operatorname{span}\{a_{i_1} \wedge \cdots \wedge a_{i_{k_A}} \wedge b_{i_{k_A+1}} \wedge \cdots \wedge b_{i_{k_A+k_B}} | 1 \leqslant i_1, \ldots, i_{k_A} \leqslant n_A, 1 \leqslant i_{k_A+1}, \ldots, i_{k_A+k_B} \leqslant n_B\} = \mathcal{H}_0$ where $\{a_i\}_{i=1}^{n_A}$ and $\{b_i\}_{i=1}^{n_B}$ are orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then v can also be written in the form $v_1^A \wedge \cdots \wedge v_{k_A}^A \wedge v_1^B \wedge \cdots \wedge v_{k_B}^B$ where $\{v_i^A\}_{i=1}^{k_A} \subset \mathcal{H}_A$ and $\{v_i^B\}_{i=1}^{k_B} \subset \mathcal{H}_B$.*

Proof. Let $k = k_A + k_B$ and $v = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ be a decomposition of v. Each v_i can be uniquely written in the form $v_i = A_i + B_i$ where $A_i \in \mathcal{H}_A$ and $B_i \in \mathcal{H}_B$. Now take a look at the terms in the expanded form of $v = (A_1 + B_1) \wedge \cdots \wedge (A_k + B_k)$. A simple observation is that two wedge products of elements of $\mathcal{H}_A$ and $\mathcal{H}_B$ are orthogonal with respect to the induced inner product if the number of factors from $\mathcal{H}_A$ (or $\mathcal{H}_B$) differs in the two products. This means that in v the term $B_1 \wedge \cdots \wedge B_k$ is orthogonal to all the other terms. But since it is orthogonal to $\mathcal{H}_0$ too it must be 0 which is equivalent to stating that the vectors $(B_i)_{i=1}^k$ are linearly dependent. After some rearranging (and possibly including a minus sign) we can assume that B_1 can be expressed in the form $\lambda_2 B_2 + \cdots + \lambda_k B_k$. Now using multilinearity and that wedge product of linearly dependent vectors is the null vector we can write

$$\begin{aligned} v &= (A_1 + B_1) \wedge \cdots \wedge (A_k + B_k) \\ &= \left(A_1 + B_1 - \sum_{i=2}^{k} \lambda_i (A_i + B_i)\right) \wedge (A_2 + B_2) \wedge \cdots \wedge (A_k + B_k) \\ &= A_1' \wedge (A_2 + B_2) \wedge \cdots \wedge (A_k + B_k) \end{aligned} \tag{A.1}$$

for some $A_1' \in \mathcal{H}_A$. Similar reasoning with the term $A_1' \wedge B_2 \wedge \cdots \wedge B_k$ shows that we can assume that B_2 can be written as a linear combination of $B_3, \ldots, B_k$ and so on finally arriving at the form $v = A_1' \wedge \cdots \wedge A_{k_A}' \wedge (A_{k_A+1} + B_{k_A+1}) \wedge \cdots \wedge (A_k + B_k)$. The number of Bs cannot be further reduced since the term containing k_B factors from $\mathcal{H}_B$ is not orthogonal to $\mathcal{H}_0$.

After expanding we see that $A_1' \wedge \cdots \wedge A_{k_A}' \wedge A_{k_A+1} \wedge \cdots \wedge A_k$ is orthogonal to the other addends and to $\mathcal{H}_0$; therefore, the factors are linearly dependent. By rearranging we can identify two cases: either A_1' or A_{k_A+1} can be expressed as a linear combination of the remaining factors. In the first case A_1' cannot be an element of $\operatorname{span}\{A_2', \ldots, A_{k_A}'\}$ since $v \neq 0$, and therefore we can find an element in $\{A_i\}_{i=k_A+1}^k$ whose coefficient in the linear expansion is nonzero leading us to the latter case. We can assume that $A_{k_A+1} = \sum_{i=1}^{k_A} \mu_i A_i' + \sum_{i=k_A+2}^{k} \mu_i A_i$ and using this expansion one can write

$$\begin{aligned} v &= A_1' \wedge \cdots \wedge A_{k_A}' \wedge (A_{k_A+1} + B_{k_A+1}) \wedge \cdots \wedge (A_k + B_k) \\ &= A_1' \wedge \cdots \wedge A_{k_A}' \wedge \left(A_{k_A+1} + B_{k_A+1} - \sum_{i=1}^{k_A} \mu_i A_i' - \sum_{i=k_A+2}^{k} \mu_i (A_i + B_i)\right) \wedge \\ &\quad \wedge (A_{k_A+2} + B_{k_A+2}) \wedge \cdots \wedge (A_k + B_k) \\ &= A_1' \wedge \cdots \wedge A_{k_A}' \wedge B_{k_A+1}' \wedge (A_{k_A+2} + B_{k_A+2}) \wedge \cdots \wedge (A_k + B_k) \end{aligned} \tag{A.2}$$

for some $B_{k_A+1}' \in \mathcal{H}_B$. Proceeding the same way we finally arrive at the form $v = A_1' \wedge \cdots \wedge A_{k_A}' \wedge B_{k_A+1}' \wedge \cdots \wedge B_k'$. □

Theorem. *Let $N \in \mathbb{N}$, $(k_i)_{i=1}^N$ and $(n_i)_{i=1}^N$ be N-tuples of positive integers, $\mathcal{H}_i^{(0)}$ a Hilbert space of dimension n_i, $\mathcal{H}_i = \bigwedge^{k_i} \mathcal{H}_i^{(0)}$ (for all $i \in \{1, \ldots, N\}$), $\mathcal{K}^{(0)} = \bigoplus_{i=1}^N \mathcal{H}_i^{(0)}$ and $\mathcal{K} = \bigwedge^k \mathcal{K}^{(0)}$ where $k = \sum_{i=1}^N k_i$. Let $\varphi : \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \to \mathcal{K}$ be the linear map defined by $(e_{1,j_1} \wedge \cdots \wedge e_{1,j_{k_1}}) \otimes \cdots \otimes (e_{N,j_{k-k_N+1}} \wedge \cdots \wedge e_{N,j_k}) \mapsto e_{1,j_1} \wedge \cdots \wedge e_{1,j_{k_1}} \wedge \cdots \wedge e_{N,j_{k-k_N+1}} \wedge \cdots \wedge e_{N,j_k}$ for some orthonormal bases $\{e_{i,j}\}_{j=1}^{k_i} \subset \mathcal{H}_i^{(0)}$. Then a vector v in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N = \mathcal{H}$ is a tensor product of decomposable vectors in $\mathcal{H}_i$ iff $\varphi(v) \in \mathcal{K}$ is decomposable.*

Proof. Since both sides of the equation defining φ is linear in all e_{i,j_l}-s, the image of $v = (v_{1,j_1} \wedge \cdots \wedge v_{1,j_{k_1}}) \otimes \cdots \otimes (v_{N,j_{k-k_N+1}} \wedge \cdots \wedge v_{N,j_k})$ is $v_{1,j_1} \wedge \cdots \wedge v_{1,j_{k_1}} \wedge \cdots \wedge v_{N,j_{k-k_N+1}} \wedge \cdots \wedge v_{N,j_k}$ which is a decomposable element of $\mathcal{K}$.

For the converse observe that by introducing the Hilbert spaces $\mathcal{K}_i^{(0)} = \mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_i^{(0)}$ and $\mathcal{K}_i = \bigwedge^{k_1+\cdots+k_i} \mathcal{K}_i^{(0)}$, for the linear injections $\varphi_i : \mathcal{K}_i \otimes \mathcal{H}_{i+1} \to \mathcal{K}_{i+1}$; $(x_1 \wedge \cdots \wedge x_{k_1+\cdots+k_i}) \otimes (y_1 \wedge \cdots \wedge y_{k_{i+1}}) \mapsto x_1 \wedge \cdots \wedge x_{k_1+\cdots+k_i} \wedge y_1 \wedge \cdots \wedge y_{k_{i+1}}$ we have $\varphi = \varphi_{N-1} \circ (\varphi_{N-2} \otimes \mathrm{id}_{\mathcal{H}_N}) \circ \cdots \circ (\varphi_1 \otimes \mathrm{id}_{\mathcal{H}_3} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_N})$. Now take a vector v in $\mathrm{ran}\varphi \subset \mathcal{K}$ that is decomposable. Using the lemma above with $\mathcal{H}_A^{(0)} = \mathcal{K}_{N-1}^{(0)}, \mathcal{H}_B^{(0)} = \mathcal{H}_N, k_A = k - k_N$ and $k_B = k_N$ we see that $v = x_1 \wedge \cdots \wedge x_{k-k_N} \wedge y_1 \wedge \cdots \wedge y_{k_N}$ for some $\{x_j\}_{j=1}^{k-k_N} \subset \mathcal{K}_{N-1}^{(0)}$ and $\{y_j\}_{j=1}^{k_N} \subset \mathcal{H}_N^{(0)}$. This means that $\varphi_{N-1}^{-1}(v) = (x_1 \wedge \cdots \wedge x_{k-k_N}) \otimes (y_1 \wedge \cdots \wedge y_{k_N})$. The first factor is in $\mathrm{ran}\varphi_{N-2}$ and hence we can apply the lemma to it, and so on, finally obtaining an N-fold tensor product of decomposable elements in $\mathcal{H}_i$. □

References

[1] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[2] Plenio M B and Virmani S 2007 *Quant. Inf. Comput.* **7** 1
[3] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2007 arXiv:quant-ph/0702225
[4] Bengtsson I and Życzkowski K 2006 *Geometry of Quantum States* (Cambridge: Cambridge University Press)
[5] Eckert K, Schliemann J, Bruss D and Lewenstein M 2002 *Ann. Phys., NY* **299** 88
[6] Li Y S, Zeng B, Liu X S and Long G L 2001 *Phys. Rev.* A **64** 054302
[7] Pakauskas R and You L 2001 *Phys. Rev.* A **64** 042310
[8] Ghirardi G C and Marinatto L 2004 *Phys. Rev.* A **70** 012109
Ghirardi G C, Marinatto L and Weber T 2002 *J. Stat. Phys.* **108** 49
[9] Schliemann J, Cirac J I, Kus M, Lewenstein M and Loss D 2001 *Phys. Rev.* A **64** 022303
Schliemann J, Loss D and MacDonald A H 2001 *Phys. Rev.* B **63** 085311
[10] Wang X and Sanders B C 2005 *J. Phys. A: Math. Gen.* **38** L67
[11] Gittings J R and Fischer A J 2002 *Phys. Rev.* A **66** 032305
[12] Lévay P, Nagy Sz and Pipek J 2005 *Phys. Rev.* A **72** 022302
[13] Lévay P and Vrana P 2008 *Phys. Rev.* A **78** 022329
[14] McCrimmon K *A Taste of Jordan Algebras Universitext* (New York: Springer)
[15] Freudenthal H 1954 *Indigationes Math.* **16** 218
Freudenthal H 1954 *Indigationes Math.* **16** 363
Freudenthal H 1955 *Indigationes Math.* **17** 151
Freudenthal H 1955 *Indigationes Math.* **17** 277
[16] Krutelevich S 2004 arXiv:math/0411104
[17] Gunaydin M, Sierra G and Townsend P K 1984 *Nucl. Phys.* B **242** 244
[18] Gunaydin M, Sierra G and Townsend P K 1983 *Phys. Lett.* B **133** 72
[19] Gunaydin M, Sierra G and Townsend P K 1985 *Nucl. Phys.* B **253** 573
[20] Ferrara S and Maldacena J M 1998 *Class. Quantum Grav.* **15** 749
[21] Ferrara S and Gunaydin M 1998 *Int J. Mod. Phys.* A **13** 2075
[22] Duff M J 2007 *Phys. Rev.* D **76** 025017
[23] Kallosh R and Linde A 2006 *Phys. Rev.* D **73** 104033
[24] Lévay P 2006 *Phys. Rev.* D **74** 024030
[25] Ferrara S and Duff M J 2007 *Phys. Rev.* D **76** 025018
[26] Lévay P 2007 *Phys. Rev.* D **75** 024024

[27] Lévay P 2007 *Phys. Rev.* D **76** 106011
[28] Ferrara S and Duff M J 2007 *Phys. Rev.* D **76** 124023
[29] Borsten L, Dahanayake D, Duff M J, Ebrahim H and Rubens W 2008 *Phys. Rev. Lett.* **100** 251602
[30] Lévay P, Saniga M and Vrana P 2008 *Phys. Rev.* D **78** 124022
[31] Borsten L 2008 *Fortschr. Phys.* **56** 842
[32] Borsten L, Dahanayake D, Duff M J, Ebrahim H and Rubens W 2008 arXiv:0809.4685
[33] Borsten L, Dahanayake D, Duff M J, Ebrahim H and Rubens W 2008 arXiv:0812.3322
[34] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev.* A **61** 052306
[35] Bellucci S, Marrani A, Orazi E and Shcherbakov A 2007 *Phys. Lett.* B **655** 185
[36] Duff M J, Liu J T and Rahmfeld J 1996 *Nucl. Phys.* B **459** 125
[37] Behrndt K, Kallosh R, Rahmfeld J, Shmakova M and Wong W K 1996 *Phys. Rev.* D **54** 6293
[38] Saraikin K and Vafa C 2008 *Class. Quantum Grav.* **25** 095007
[39] Brody D C, Gustavsson A C T and Hughston L P 2007 *J. Phys.: Conf. Ser.* **67** 012044
[40] Hodge W V D and Pedoe D 1994 *Methods of Algebraic Geometry* vol II (Cambridge: Cambridge University Press)
[41] Kasman A, Pedings K, Reiszl A and Shiota T 2005 arXiv:math/0510093
[42] Wong A and Christensen N 2001 *Phys. Rev.* A **63** 044301
[43] Luque J-G and Thibon J-Y 2003 *Phys. Rev.* A **67** 042303